Improper Integrals

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Improper Integrals

Definition

An integral is improper if either

1. the interval of integration is infinitely long or
2. if the function has singularities in the interval of integration.

One cannot apply numerical methods like LEFT or RIGHT sums to approximate the value of such integrals.

Examples

1 The integral $\int_0^{\infty} e^{-x}\, dx$ is improper because the interval of integration is infinitely long.

2 $\int_0^1 \frac{1}{x^2} dx$ is improper because the integrand has a singularity.

3 $\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}} dx$ is improper for two reasons: infinitely long interval of integration and a singularity of the integrand.

Definition of Improper Integrals

Definition

Assume that the function f takes finite values in the interval $[a,\infty)$.

If the limit $\lim_{b\to\infty}\int_a^b f(x)dx$ exists and is finite, then we say that the the improper integral of the function f over the interval $[a,\infty)$ converges.

In this case we set

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx.$$

The improper integral $\int_a^\infty f(x)dx$ diverges if the limit $\lim_{b\to\infty}\int_a^b f(x)dx$ does not exist or is not finite.

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big]_0^b$$

$$= \lim_{b\to\infty} -e^{-b}-(-e^0) = 1$$

Singularities in the Interval of Integration

Definition

Assume that the function f has a singularity at the point $x = a$ but that f otherwise takes finite values. Then

$$\int_a^b f(x)dx = \lim_{\alpha \to a+} \int_\alpha^b f(x)dx.$$

If this limit exists and is finite, then the improper integral $\int_a^b f(x)dx$ converges. Otherwise it diverges.

Improper integrals of functions f having a singularity at b or somewhere inside the interval of integration are defined in a similar way as limits of ordinary integrals over intervals which do not contain the singular point.

Example

$$\int_0^1 \frac{1}{x} dx = \lim_{\alpha \to 0+} \int_\alpha^1 \frac{1}{x} dx = \lim_{\alpha \to 0+} \ln(1) - \ln(\alpha) = \infty.$$

The improper integral diverges.

Generalizations

The previous definitions generalize to the cases where one of the end-points of the interval of integration is negative infinity or a singular point of the function is contained in the interval of integration.

Examples

1

$$\int_{-\infty}^{-1}\frac{1}{x^2}dx = \lim_{\alpha\to-\infty}\int_{\alpha}^{-1}\frac{1}{x^2}dx = \lim_{\alpha\to-\infty}-\frac{1}{x}\Bigg]_{\alpha}^{-1} = \lim_{\alpha\to-\infty}1-\frac{1}{\alpha} = 1.$$

This integral converges.

2

$$\int_{-1}^{1}\frac{1}{x^2}dx = \lim_{\beta\to0-}\int_{-1}^{\beta}\frac{1}{x^2}dx + \lim_{\alpha\to0+}\int_{\alpha}^{1}\frac{1}{x^2}dx$$

$$= \lim_{\beta\to0-}-\frac{1}{x}\Bigg]_{-1}^{\beta} + \lim_{\alpha\to0+}-\frac{1}{x}\Bigg]_{\alpha}^{1} = \infty+\infty=\infty$$

This integral diverges.

Basic Improper Integrals

1 $\int_1^\infty x^p dx$ converges for $p < -1$ and diverges otherwise

2 $\int_1^\infty \frac{1}{x^p} dx$ converges for $p > 1$ and diverges otherwise

3 $\int_0^1 x^p dx$ converges for $p > -1$ and diverges otherwise

4 $\int_0^1 \frac{1}{x^p} dx$ converges for $p < 1$ and diverges otherwise

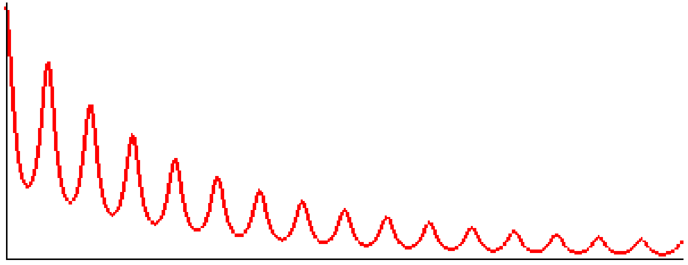
5 $\int_0^\infty e^{ax} dx$ converges for $a < 0$ and diverges otherwise

Clearly (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4). To prove these results is a straightforward computation.

Convergence of Improper Integrals

Often it is not possible to compute the limit defining a given improper integral directly. In order to find out whether such an integral converges or not one can try to compare the integral to a known integral of which we know that it either converges or diverges.

$\int_0^{\infty} \frac{3}{(1+2\sin^2(4\pi x))(1+x^2)}dx$ is an example of such an improper integral. The graph of the function to be integrated is shown in the picture.



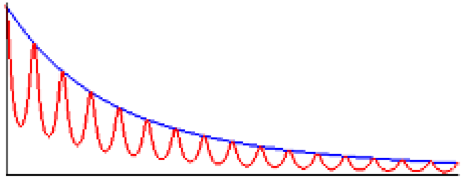
Idea of the Comparison Theorem

Observe that the function to be integrated satisfies

$$0 < \frac{3}{(1+2\sin^2(4\pi x))(1+x^2)} \le \frac{3}{1+x^2}$$

for all $x \ge 0$. The following graph illustrates this observation.

The blue curve is the graph of the function $\frac{3}{1+x^2}$ while the red curve is the graph of the function to be integrated.



The improper integral $\int_0^\infty \frac{3}{(1+2\sin^2(4\pi x))(1+x^2)}dx$ converges if the area under the red curve is finite. We show that this is true by showing that the area under the blue curve is finite. Since the area under the red curve is smaller than the area under the blue curve, it must then also be finite. This means that the complicated improper integral converges.

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Examples (1)

To show that the area under the blue curve in the previous figure is finite, compute as follows:

$$\int_0^\infty \frac{3}{1+x^2}dx = \lim_{b\to\infty}\int_0^b \frac{3}{1+x^2}dx = \lim_{b\to\infty} 3\arctan(x)\Big|_0^b = \frac{3\pi}{2}.$$

This means that the improper integral $\int_0^\infty \frac{3}{1+x^2}dx$ converges.

Hence also the improper integral $\int_0^\infty \frac{3}{(1+2\sin^2(4\pi x))(1+x^2)}dx$ converges.

Comparison Theorem

Theorem

Let $a,b \in \mathbb{R} \cup \{\infty, -\infty\}$, $a < b$. Assume that the functions f and g satisfy $0 \le \mathrm{f}(x) \le \mathrm{g}(x)$ for all x, $a < x < b$. Assume also that the integral $\int_a^b \mathrm{f}(x)dx$ is improper.

1) If the improper integral $\int_a^b \mathrm{g}(x)dx$ converges, then also the improper integral $\int_a^b \mathrm{f}(x)dx$ converges, and $0 \le \int_a^b \mathrm{f}(x)dx \le \int_a^b \mathrm{g}(x)dx$.

2) If the improper integral $\int_a^b \mathrm{f}(x)dx$ diverges, then also the improper integral $\int_a^b \mathrm{g}(x)dx$ diverges.

Remark The integral $\int_a^b \mathrm{f}(x)dx$ is improper if either $a = \infty, b = \infty$ or the function f has singularities in the interval of integration.

Normal Distribution Function

The improper integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ has important applications. It is related to the normal distribution, an important concept in statistics. We need to understand why the improper intergal converges.

To show the convergence write $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} e^{-x^2}\,dx$.

The integral $\int_{-1}^{1} e^{-x^2}\,dx$ is an ordinary definite integral which has a finite value.

To study the convergence of the integral $\int_{1}^{\infty} e^{-x^2}\,dx$ observe that for $x \geq 1,\ -x^2 \leq -x$. Hence also $0 < e^{-x^2} \leq e^{-x}$ for $x \geq 1$. The improper integral $\int_{1}^{\infty} e^{-x}\,dx$ converges.

This has been shown earlier. Hence we conclude that the integral $\int_{1}^{\infty} e^{-x^2}\,dx$ converges. The same argument shows that also $\int_{-\infty}^{-1} e^{-x^2}\,dx$ converges. Hence the integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges.

Examples (2)

Problem

The integral $\int_0^1 \frac{1}{\sin(x)} dx$ is improper because $\frac{1}{\sin(x)}$ is undefined at $x = 0$. Does this integral converge?

Solution

To study the convergence of this integral recall that for

$$x > 0, \ \sin(x) < x. \text{ If } 0 < x < 1,$$

then also $\sin(x) > 0$. We conclude that

$$\forall x \in (0,1) : 0 < \sin(x) < x \Rightarrow \forall x \in (0,1) : 0 < \frac{1}{x} < \frac{1}{\sin(x)}.$$

The improper integral $\int_0^1 \frac{1}{x} dx$ is one of the basic types of improper integrals. We know that it diverges. Hence we conclude, by the Comparison Theorem, that also the integral $\int_0^1 \frac{1}{\sin(x)} dx$ diverges.

Examples (3)

Problem

Does the improper integral $\int_1^\infty \frac{x^3+1}{x^4+1}dx$ converge?

Heuristic Approach

To study the converges (or not) of this integral observe that for large values of x, $\frac{x^3+1}{x^4+1} \approx \frac{x^3}{x^4} = \frac{1}{x}$. The improper integral $\int_1^\infty \frac{1}{x}dx$ diverges. Hence also the improper integral $\int_1^\infty \frac{x^3+1}{x^4+1}$ diverges.

Examples (4)

Problem

Does the improper integral $\int_1^\infty \frac{x^3+1}{x^4+1}dx$ converge?

Rigorous Solution

The divergence of the integral can be justified by the Comparison Theorem in the following way.

$x>1 \Rightarrow x^4>1 \Rightarrow 2x^4 = x^4+x^4 > x^4+1.$ This implies

$x>1 \Rightarrow \frac{1}{x^4+1} > \frac{1}{2x^4}.$ Hence $x>0 \Rightarrow \frac{x^3+1}{x^4+1} > \frac{x^3}{x^4+1} > \frac{x^3}{2x^4} = \frac{1}{2x} > 0.$

Next recall that $\int_1^\infty \frac{1}{2x}dx = \lim_{b\to\infty}\frac{1}{2}\int_1^b \frac{1}{x}dx = \lim_{b\to\infty}\frac{\ln(b)}{2} = \infty$, i.e., that the integral $\int_1^\infty \frac{1}{2x}dx$ diverges. Comparison Theorem now tells us that also the integral $\int_1^\infty \frac{x^3+1}{x^4+1}dx$ diverges.

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FAQ

Example: The Gamma Function

Problem

The gamma function is defined for $x > 0$ by setting

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$. Show that this integral converges.

Solution

The integral defining the gamma function is improper because the interval of integration extends to the infinity. If $0 < x < 1$, the integral is also improper because then the function to be integrated has a singularity at $x = 0$, the left end point of the interval of integration.

Case 1.
$0 < t < 1$.

Observe first that the integral $\int_0^1 t^p dt$ converges if $p > -1$.

$$\int_0^1 t^p dt = \lim_{a \to 0+} \int_a^1 t^p dt = \lim_{a \to 0+} \left[\frac{t^{p+1}}{p+1} \right]_a^1 = \lim_{a \to 0+} \left(\frac{1}{p+1} - \frac{a^{p+1}}{a+1} \right) = \frac{1}{p+1}$$

This computation requires the assumption that $p > -1$, i.e., that $p + 1 > 0$. This allows you to conclude that $a^{p+1} \to 0$ as $a \to 0$.

Example: The Gamma Function

Problem

Solution (cont'd)

The gamma function is defined for $x > 0$ by setting

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$. Show that this integral converges.

Next observe that, if $t > 0$, $t^{x-1} e^{-t} < t^{x-1}$.

Case 1. ($0 < t < 1$)

Hence the integral $\int_0^1 t^{x-1} e^{-t}\, dt$ converges by the Comparison Theorem and by the fact that $\int_0^1 t^p dt$ converges for $p > -1$.

Case 2. $t > 1$.

To show the convergence of the integral $\int_1^\infty t^{x-1} e^{-t}\, dt$ use the fact that $\lim_{t\to\infty} t^{x-1} e^{-\frac{t}{2}} = 0$. This holds for all values of x. Hence there is a number b_x such that $t^{x-1} e^{-t/2} < 1$ for $t > b_x$.

This means that $t^{x-1} e^{-t} = t^{x-1} e^{-t/2} e^{-t/2} < e^{-t/2}$ for $t > b_x$.

Hence the integral $\int_{b_x}^\infty t^{x-1} e^{-t}\, dt$ converges by the Comparison Theorem since the integral $\int_{b_x}^\infty e^{-t/2}\, dt$ converges as can be seen by a direct computation.

Example: The Gamma Function

Problem

The gamma function is defined for $x > 0$ by setting

$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$. Show that this integral converges.

Solution

Let $x > 0$ be fixed.

We split the improper integral defining the Gamma function to three integrals as follows:

Conclusion

$$\int_0^\infty t^{x-1} e^{-t}\, dt = \int_0^1 t^{x-1} e^{-t}\, dt + \int_1^{b_x} t^{x-1} e^{-t}\, dt + \int_{b_x}^\infty t^{x-1} e^{-t}\, dt$$

1 $\int_0^1 t^{x-1} e^{-t}\, dt$ converges by part 1. Here we needed to assume that $x > 0$.

2 $\int_1^{b_x} t^{x-1} e^{-t}\, dt$ is an ordinary integral.

3 $\int_{b_x}^\infty t^{x-1} e^{-t}\, dt$ converges by part 2.

We conclude that the integral $\int_0^\infty t^{x-1} e^{-t}\, dt$ converges. $\blacksquare$